The Hassenpflug matrix tensor notation

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Abstract

This is a sample document to illustrate the typesetting of vectors, matrices and tensors according to the matrix tensor notation of Hassenpflug[1, 2]. The first section describes the bare basics of the notation and please note that there is much more to the notation than the little bit described here. The second and third sections are applications of the notation in rotation kinematics.

Keywords: *vector, matrix, tensor, notation*

N.B. — *This document is neither a guide nor a reference document for the Hassenpflug notation. For any reference to the material in §1 (excluding equation 1.9), please cite the original copyrighted articles [1, 2].*

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1 Hassenpflug matrix tensor notation

1.1 Basic vector Notation

All vectors are in the 3-dimensional Euclidean space $\mathbb{R}^3$ and tensors in $\mathbb{R}^{3\times3}$. Any other vector space will be explicitly stated. The rest of this section lists the basic definitions of the notation of *Hassenpflug*[1, 2]

$$\textit{Physical vector:}\qquad \vec{\boldsymbol{v}} \equiv \vec{\boldsymbol{e}}_1\, v_1 + \vec{\boldsymbol{e}}_2\, v_2 + \vec{\boldsymbol{e}}_3\, v_3 \tag{1.1}$$

The physical vector is the general representation of a vector in any coordinate system. The unit vectors $\vec{\boldsymbol{e}}_i$, $(i = 1, 2, 3)$, define the direction of the axes in a right-handed orthogonal Cartesian system. The components, $\vec{\boldsymbol{e}}_i\, v_i$, are the components of the vector and the scalar quantities, v_i, the elements of the vector.

$$\textit{Column vector:}\qquad \overline{\boldsymbol{v}}^a \equiv \begin{bmatrix} v_{a1} \\ v_{a2} \\ v_{a3} \end{bmatrix} \tag{1.2}$$

The column matrix of the elements of a vector is called a column vector and is the algebraic representation of a vector. The bar above the symbol of the vector indicates a column vector and the superscript (a) the index of the specific coordinate system in which the elements of the vector are expressed.

$$\textit{Row vector:}\qquad \underline{\boldsymbol{v}}_a \equiv \left[\overline{\boldsymbol{v}}^a\right]^{\mathsf{T}} = \begin{bmatrix} v_{a1} & v_{a2} & v_{a3} \end{bmatrix} \tag{1.3}$$

The row matrix of the elements of a vector is called a row vector. The bar below the symbol of the vector indicates a row vector and the subscript (a) indicates the index of the specific coordinate system in which the elements of the vector are expressed. It is important to note that in general $\left[\overline{\boldsymbol{v}}^a\right]^{\mathsf{T}} = \underline{\boldsymbol{v}^{\mathsf{T}}}_a$ for skew and curved coordinates, see *Hassenpflug*[2]. The format in (1.3) without the transpose sign is only valid in Cartesian coordinates.

$$\textit{Norm:}\qquad \|\vec{\boldsymbol{v}}\| \equiv v, \tag{1.4a}$$

$$\|\overline{\boldsymbol{v}}\| \equiv v \equiv \sqrt{\underline{\boldsymbol{v}}\cdot\overline{\boldsymbol{v}}} = \sqrt{v_1^2 + v_2^2 + v_3^2} \tag{1.4b}$$

The norm of a vector is the algebraic size or length of the vector. The second equation, (1.4b), in element form, is only valid in Cartesian coordinates or Euclidean space.

$$\textit{Scalar, dot or inner product:}\qquad \vec{\boldsymbol{v}} \bullet \vec{\boldsymbol{u}} \equiv \underset{\rightarrow}{\underline{\boldsymbol{v}}}\cdot\vec{\boldsymbol{u}} = v\,u\,\cos\varphi, \tag{1.5a}$$

$$\overline{\boldsymbol{v}} \bullet \overline{\boldsymbol{u}} \equiv \underline{\boldsymbol{v}}\cdot\overline{\boldsymbol{u}} = v_1u_1 + v_2u_2 + v_3u_3 \tag{1.5b}$$

The scalar product of two vectors results in a scalar. The angle φ is the angle in space between $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{u}}$.

Dyad or outer product:

$$\overline{\boldsymbol{v}} \circ \overline{\boldsymbol{u}} \equiv \overline{\boldsymbol{v}} \cdot \underline{\boldsymbol{u}} = \begin{bmatrix} v_1 u_1 & v_1 u_2 & v_1 u_3 \\ v_2 u_1 & v_2 u_2 & v_2 u_3 \\ v_3 u_1 & v_3 u_2 & v_3 u_3 \end{bmatrix} \tag{1.6}$$

The dyad or outer product of two vectors results in a square matrix. There exists a well-defined algebra for dyads. It is sometimes convenient to handle second-rank Cartesian tensors such as inertia tensors as a linear polynomial of dyads, called a dyadic.

Vector or cross product:

$$\begin{aligned} \vec{\boldsymbol{v}} \times \vec{\boldsymbol{u}} \equiv (v_2 u_3 - v_3 u_2)\, &\vec{\boldsymbol{e}}_1 \\ + (v_3 u_1 - v_1 u_3)\, &\vec{\boldsymbol{e}}_2 \\ + (v_1 u_2 - v_2 u_1)\, &\vec{\boldsymbol{e}}_3, \end{aligned} \tag{1.7a}$$

$$\|\vec{\boldsymbol{a}} \times \vec{\boldsymbol{c}}\| = v\, u \sin\varphi \tag{1.7b}$$

The cross product of the two vector $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{u}}$ results in a vector perpendicular to both $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{u}}$. This operation is only defined in 3-dimensional Cartesian space. The angle φ is the angle in space between $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{u}}$.

The cross product can also be defined in terms of a matrix-vector operation $\overline{\boldsymbol{v}} \times \overline{\boldsymbol{u}} \equiv \underline{\overline{\widetilde{\boldsymbol{v}}}} \cdot \overline{\boldsymbol{u}}$

Cross product tensor:

$$\underline{\overline{\widetilde{\boldsymbol{v}}}} \equiv \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \tag{1.8}$$

Various identities for the cross product tensor can be verified. These identities will be extensively used throughout this article.

$$\begin{array}{lll} \left[\underline{\overline{\widetilde{\boldsymbol{v}}}}\right]^{\mathsf{T}} = -\underline{\overline{\widetilde{\boldsymbol{v}}}} & \left[\underline{\overline{\widetilde{\boldsymbol{v}}}}\right]^2 = \overline{\boldsymbol{v}} \cdot \underline{\boldsymbol{v}} - v^2 \underline{\overline{\boldsymbol{I}}} & \underline{\overline{\widetilde{\boldsymbol{v} + \boldsymbol{u}}}} = \underline{\overline{\widetilde{\boldsymbol{v}}}} + \underline{\overline{\widetilde{\boldsymbol{u}}}} \\ \underline{\overline{\widetilde{\boldsymbol{v}}}} \cdot \overline{\boldsymbol{u}} = -\underline{\overline{\widetilde{\boldsymbol{u}}}} \cdot \overline{\boldsymbol{v}} & \left[\underline{\overline{\widetilde{\boldsymbol{v}}}}\right]^3 = -v^2 \underline{\overline{\widetilde{\boldsymbol{v}}}} & \underline{\overline{\widetilde{\underline{\overline{\widetilde{\boldsymbol{v}}}} \cdot \overline{\boldsymbol{u}}}}} = \underline{\overline{\widetilde{\boldsymbol{v}}}} \cdot \underline{\overline{\widetilde{\boldsymbol{u}}}} - \underline{\overline{\widetilde{\boldsymbol{u}}}} \cdot \underline{\overline{\widetilde{\boldsymbol{v}}}} \end{array} \tag{1.9}$$

with $\underline{\overline{\boldsymbol{I}}}$ the 3×3 *identity matrix.*

$$\underline{\overline{\boldsymbol{I}}} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.10}$$

1.2 Vector Transformations

In this section only a basic overview of vector rotations and transformations is given to establish the basic nomenclature and definitions. For a more in-depth discussion refer to *Hassenpflug*[1].

Consider two Cartesian axis systems denoted by s and r as shown in Fig. 1.1(a) on the following page. From the general definition of a vector, (1.1), follows

$$\vec{\boldsymbol{v}} = \begin{bmatrix} \vec{\boldsymbol{e}}_{s1} & \vec{\boldsymbol{e}}_{s2} & \vec{\boldsymbol{e}}_{s3} \end{bmatrix} \cdot \begin{bmatrix} v_{s1} \\ v_{s2} \\ v_{s3} \end{bmatrix} = \underline{\vec{\boldsymbol{E}}}_s \cdot \overline{\boldsymbol{v}}_s \tag{1.11}$$

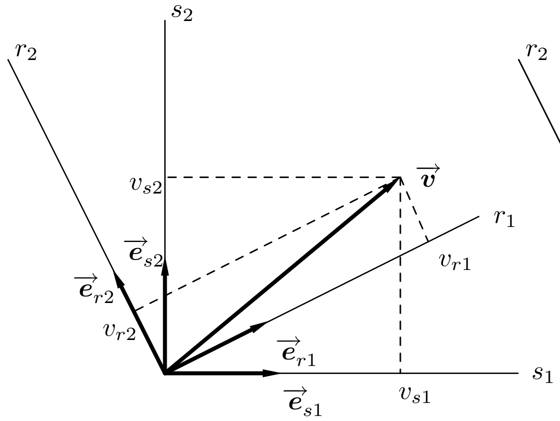

Figure 1.1(a): Vector transformation

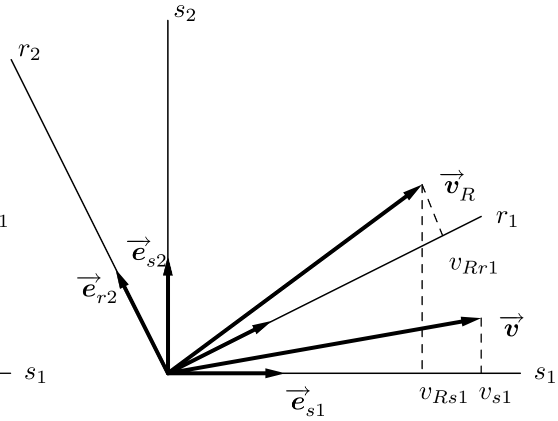

Figure 1.1(b): Vector rotations

The quantity, $\underline{\vec{\boldsymbol{E}}}_s = [\vec{\boldsymbol{e}}_{s1}\ \vec{\boldsymbol{e}}_{s2}\ \vec{\boldsymbol{e}}_{s3}]$, is the base of the axis system denoted by s. It consists of the three orthogonal vectors parallel to the axes. From the outer product (1.6) follows for the inverse of base $\underline{\vec{\boldsymbol{E}}}_s$:

$$\left[\underline{\vec{\boldsymbol{E}}}_s\right]^{\mathsf{T}} \cdot \underline{\vec{\boldsymbol{E}}}_s = \underrightarrow{\boldsymbol{E}}^s \cdot \underline{\vec{\boldsymbol{E}}}_s = \underline{\boldsymbol{I}} \quad \Rightarrow \quad \left[\underline{\vec{\boldsymbol{E}}}_s\right]^{\mathsf{T}} = \left[\underline{\vec{\boldsymbol{E}}}_s\right]^{-1} = \underrightarrow{\boldsymbol{E}}^s \tag{1.12}$$

We can repeat the procedure of (1.11) for the vector $\vec{\boldsymbol{v}}$ in terms of base $\underline{\vec{\boldsymbol{E}}}_r$. The relationship of the elements of vector $\vec{\boldsymbol{v}}$ in terms of base $\underline{\vec{\boldsymbol{E}}}_s$ and base $\underline{\vec{\boldsymbol{E}}}_r$ is then

$$\vec{\boldsymbol{v}} = \underline{\vec{\boldsymbol{E}}}_r \cdot \overline{\boldsymbol{v}}^r = \underline{\vec{\boldsymbol{E}}}_s \cdot \overline{\boldsymbol{v}}^s \quad \Rightarrow \quad \begin{cases} \overline{\boldsymbol{v}}^s & = \underrightarrow{\boldsymbol{E}}^s \cdot \underline{\vec{\boldsymbol{E}}}_r = \underline{\overline{\boldsymbol{E}}}_r^s \cdot \overline{\boldsymbol{v}}^r \\ \overline{\boldsymbol{v}}^r & = \underrightarrow{\boldsymbol{E}}^r \cdot \underline{\vec{\boldsymbol{E}}}_s = \underline{\overline{\boldsymbol{E}}}_s^r \cdot \overline{\boldsymbol{v}}^s \end{cases} \tag{1.13}$$

The matrix quantities $\underline{\overline{\boldsymbol{E}}}_r^s$ and $\underline{\overline{\boldsymbol{E}}}_s^r$ are then the transformation matrices of the components of a vector between the two bases $\underline{\vec{\boldsymbol{E}}}_s$ and $\underline{\vec{\boldsymbol{E}}}_r$. The columns of the transformation matrix $\underline{\overline{\boldsymbol{E}}}_r^s$ are the elements of the unit vector $\vec{\boldsymbol{e}}_{s_i}$ expressed in base $\underrightarrow{\boldsymbol{E}}_s$ and the rows are the unit vectors $\underrightarrow{\boldsymbol{e}}^{s_j}$ expressed in base $\underline{\vec{\boldsymbol{E}}}_r$.

$$\underline{\overline{\boldsymbol{E}}}_r^s = \left[\overline{\boldsymbol{e}}_{r1}^s \quad \overline{\boldsymbol{e}}_{r2}^s \quad \overline{\boldsymbol{e}}_{r3}^s\right] = \begin{bmatrix} \underline{\boldsymbol{e}}_r^{s1} \\ \underline{\boldsymbol{e}}_r^{s2} \\ \underline{\boldsymbol{e}}_r^{s3} \end{bmatrix} \tag{1.14}$$

The properties of the transformation matrix are well-known, for example

$$\left[\underline{\overline{\boldsymbol{E}}}_r^s\right]^{\mathsf{T}} = \left[\underline{\overline{\boldsymbol{E}}}_r^s\right]^{-1} = \underline{\overline{\boldsymbol{E}}}_s^r \tag{1.15}$$

1.3 Vector rotations

Consider the case of a vector in space with initial position $\vec{\boldsymbol{v}}$. The vector is rotated to a new position in space, $\vec{\boldsymbol{v}}_R$. Define the rotation tensor operation then as

$$\vec{\boldsymbol{v}}_R = \underrightarrow{\vec{\boldsymbol{R}}} \cdot \vec{\boldsymbol{v}} \tag{1.16}$$

If the operation is applied to the rotation of all the direction vectors of a base $\overrightarrow{\underline{\boldsymbol{E}}}_s$ to a new rotated base $\overrightarrow{\underline{\boldsymbol{E}}}_r$, then

$$\overrightarrow{\underline{\boldsymbol{E}}}_r = \underrightarrow{\overrightarrow{\boldsymbol{R}}}\cdot\overrightarrow{\underline{\boldsymbol{E}}}_s \tag{1.17}$$

or

$$\overline{\underline{\boldsymbol{E}}}^s_r = \underrightarrow{\overline{\boldsymbol{E}}}^s\cdot\underrightarrow{\overrightarrow{\boldsymbol{R}}}\cdot\overrightarrow{\underline{\boldsymbol{E}}}_s = \overline{\underline{\boldsymbol{R}}}^s_s \tag{1.18}$$

With reference to Fig. 1.1(b) on the page before, consider the case of a vector fixed in a rotating base $\overrightarrow{\underline{\boldsymbol{E}}}_r$ with initial position $\overrightarrow{\boldsymbol{v}}$ and final position after a rotation of $\overrightarrow{\boldsymbol{v}}_R$. If the initial orientation of $\overrightarrow{\underline{\boldsymbol{E}}}_r$ corresponds with that of $\overrightarrow{\underline{\boldsymbol{E}}}_s$, the numerical values of the components of $\overline{\boldsymbol{v}}^s$ and $\overline{\boldsymbol{v}}^r_R$ are equal. From the transformation of $\overrightarrow{\boldsymbol{v}}_R$ it then follows that

$$\overline{\boldsymbol{v}}^s_R = \overline{\underline{\boldsymbol{E}}}^s_r\cdot\overline{\boldsymbol{v}}^r_R = \overline{\underline{\boldsymbol{R}}}^s_s\cdot\overline{\boldsymbol{v}}^s \tag{1.19}$$

If the rotation matrix is transformed between bases, then

$$\overline{\underline{\boldsymbol{R}}}^r_r = \overline{\underline{\boldsymbol{E}}}^r_s\cdot\overline{\underline{\boldsymbol{R}}}^s_s\cdot\overline{\underline{\boldsymbol{E}}}^s_r = \overline{\underline{\boldsymbol{R}}}^s_s \tag{1.20}$$

The rotation matrix is therefore identical in terms of both bases and we can denote it without the base indices, except when there is more than one rotation. The rotation matrix between bases $\overrightarrow{\underline{\boldsymbol{E}}}_s$ and $\overrightarrow{\underline{\boldsymbol{E}}}_r$ in terms of the transformation matrix is given by

$$\overline{\underline{\boldsymbol{R}}} = \overline{\underline{\boldsymbol{E}}}^s_r \tag{1.21}$$

$$\left[\overline{\underline{\boldsymbol{R}}}\right]^{-1} = \left[\overline{\underline{\boldsymbol{R}}}\right]^{\mathsf{T}} = \overline{\underline{\boldsymbol{E}}}^r_s \tag{1.22}$$

2 Rotation kinematics

2.1 The rotation matrix (Rodriguez formula)

Euler's theorem states that the most general displacement of a rigid body with one point fixed is equivalent to a single rotation about some axis through that point. With reference to Fig. 2.1 on the following page, consider a vector with initial position $\overrightarrow{\boldsymbol{v}}$. The vector is rotated about an axis defined by the unit vector $\overrightarrow{\boldsymbol{a}}$, through an angle ϑ. The vector after rotation is denoted by $\overrightarrow{\boldsymbol{v}}_R$. From the geometry in Fig. 2.1 on the next page, it can be shown (e.g., *Argyris*[3]) for the vector components in terms of the stationary base $\overrightarrow{\underline{\boldsymbol{E}}}_s$ that

$$\overline{\boldsymbol{v}}^s_R = \overline{\boldsymbol{v}}^s + \sin\vartheta\,(\overline{\boldsymbol{a}}^s\times\overline{\boldsymbol{v}}^s) + (1-\cos\vartheta)\left(\overline{\boldsymbol{a}}^s\times(\overline{\boldsymbol{a}}^s\times\overline{\boldsymbol{v}}^s)\right) \tag{2.1a}$$

$$= \left[\overline{\underline{\boldsymbol{I}}} + \sin\vartheta\,\overline{\overline{\underline{\boldsymbol{a}}}}^s_s + (1-\cos\vartheta)\,\overline{\overline{\underline{\boldsymbol{a}}}}^s_s\cdot\overline{\overline{\underline{\boldsymbol{a}}}}^s_s\right]\cdot\overline{\boldsymbol{v}}^s \tag{2.1b}$$

Equation (2.1b) was obtained from (2.1a) with the aid of the cross product tensor (1.8) while $\overline{\underline{\boldsymbol{I}}}$ is the 3×3 unit matrix.

By comparing (2.1b) with (1.20), the general format of the rotation matrix for a rotation through an angle ϑ about an axis $\overline{\boldsymbol{a}}^s$ fixed in base $\overrightarrow{\underline{\boldsymbol{E}}}_s$ is given by

$$\overline{\underline{\boldsymbol{R}}} = \overline{\underline{\boldsymbol{I}}} + 2\cos\tfrac{\vartheta}{2}\,\sin\tfrac{\vartheta}{2}\,\overline{\overline{\underline{\boldsymbol{a}}}}^s_s + 2\sin^2\tfrac{\vartheta}{2}\,\overline{\overline{\underline{\boldsymbol{a}}}}^s_s\cdot\overline{\overline{\underline{\boldsymbol{a}}}}^s_s \tag{2.2}$$

$$\overline{\underline{\boldsymbol{R}}}^{\mathsf{T}} = \overline{\underline{\boldsymbol{I}}} - 2\cos\tfrac{\vartheta}{2}\,\sin\tfrac{\vartheta}{2}\,\overline{\overline{\underline{\boldsymbol{a}}}}^s_s + 2\sin^2\tfrac{\vartheta}{2}\,\overline{\overline{\underline{\boldsymbol{a}}}}^s_s\cdot\overline{\overline{\underline{\boldsymbol{a}}}}^s_s \tag{2.3}$$

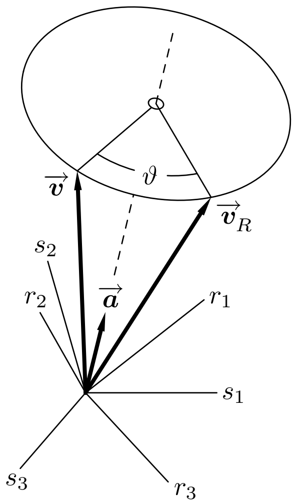

Figure 2.1: General vector rotation

Equation 2.2 is also known as the *Rodriguez formula*. The equations were rewritten in terms of $\vartheta/2$ for the convenience of definitions that follow later in the article.

If $\vec{\boldsymbol{v}}$ is fixed to a rotating base $\underline{\overrightarrow{\boldsymbol{E}}}_r$, with $\overline{\boldsymbol{v}}^s = \overline{\boldsymbol{v}}_R^r$ (see Fig. 1.1(a) and 1.1(b) on page 4), then $\underline{\overline{\boldsymbol{E}}}_r^s$ is the transformation matrix from base $\underline{\overrightarrow{\boldsymbol{E}}}_r$ to base $\underline{\overrightarrow{\boldsymbol{E}}}_s$ and

$$\underline{\overline{\boldsymbol{E}}}_r^s = \underline{\overline{\boldsymbol{R}}} \tag{2.4}$$

$$\underline{\overline{\boldsymbol{E}}}_s^r = \underline{\overline{\boldsymbol{R}}}^{\mathsf{T}} \tag{2.5}$$

Note for the transformation of the cross product tensor associated with the rotation axis, is $\underline{\overline{\tilde{\boldsymbol{a}}}}_s^s = \underline{\overline{\tilde{\boldsymbol{a}}}}_r^r = \underline{\overline{\tilde{\boldsymbol{a}}}}$, because the components are identical in both the bases. In the rest of this article the basis reference indexes for $\underline{\overline{\tilde{\boldsymbol{a}}}}$ are not shown except where a distinction must be made between two different rotations.

For numerical purposes (2.2) can be written as a single matrix. Let $c = \cos\vartheta$ and $s = \sin\vartheta$, then the rotation or transformation matrix is given by

$$\underline{\overline{\boldsymbol{R}}} = \underline{\overline{\boldsymbol{E}}}_r^s = \begin{bmatrix} a_1^2(1-c)+c & a_1a_2(1-c)-a_3s & a_1a_3(1-c)+a_2s \\ a_1a_2(1-c)+a_3s & a_2^2(1-c)+c & a_2a_3(1-c)-a_1s \\ a_1a_3(1-c)-a_2s & a_2a_3(1-c)+a_1s & a_3^2(1-c)+c \end{bmatrix} \tag{2.6}$$

It is frequently necessary to find the rotation axis $\vec{\boldsymbol{a}}$ and rotation angle ϑ for a known transformation matrix, $\underline{\overline{\boldsymbol{E}}}_r^s = E_{ij}$. From (2.6) various relationships can be deducted. Two of the more important ones are

$$2\cos\vartheta = E_{11} + E_{22} + E_{33} - 1 \tag{2.7}$$

$$2\sin\vartheta\,\overline{\boldsymbol{a}} = \begin{bmatrix} E_{32} - E_{23} \\ E_{13} - E_{31} \\ E_{21} - E_{12} \end{bmatrix} \tag{2.8}$$

When $\vartheta \approx \pi$ equation (2.8) cannot be used to find $\overline{\boldsymbol{a}}$. Another, more general, approach is to consider the characteristic polynomial of $\underline{\overline{\boldsymbol{E}}}_r^s$.

$$\det\left[\underline{\overline{\boldsymbol{E}}}_r^s - \lambda\underline{\overline{\boldsymbol{I}}}\right] = (\lambda^2 + 2\lambda\cos\vartheta + 1)(1-\lambda) = 0 \tag{2.9}$$

It leads to the eigenvalues $\lambda = \mathrm{e}^{\mathrm{i}\vartheta}, \, \mathrm{e}^{-\mathrm{i}\vartheta}, \, 1$. It can therefore be stated that $\lambda = 1$ is always an eigenvalue of $\underline{\overline{\boldsymbol{E}}}_r^s$ and that an eigenvector or axis $\overline{\boldsymbol{a}} = \overline{\boldsymbol{a}}^s = \overline{\boldsymbol{a}}^r$ exists that is unchanged by the rotation. The rotation axis can be obtained with a numerical method by solving the eigenvector problem $\underline{\overline{\boldsymbol{E}}}_r^s \cdot \overline{\boldsymbol{a}} = \overline{\boldsymbol{a}}$.

2.2 Angular velocity

Define the vectors $\overline{\boldsymbol{x}}^s$ and $\dot{\overline{\boldsymbol{x}}}^s = \mathrm{d}\overline{\boldsymbol{x}}^s/\mathrm{d}t$ as the position and velocity of a particle or point with components in terms of a static base $\underline{\overrightarrow{\boldsymbol{E}}}_s$, while $\overline{\boldsymbol{x}}^r$ and $\dot{\overline{\boldsymbol{x}}}^r$ are the position and apparent velocity in terms of a rotating base $\underline{\overrightarrow{\boldsymbol{E}}}_r$.

$$\overline{\boldsymbol{x}}^s = \underline{\overline{\boldsymbol{E}}}_r^s \cdot \overline{\boldsymbol{x}}^r \tag{2.10}$$

and

$$\dot{\overline{\boldsymbol{x}}}^s = \underline{\overline{\boldsymbol{E}}}_r^s \cdot \left[\dot{\overline{\boldsymbol{x}}}^r + \underline{\overline{\boldsymbol{E}}}_s^r \cdot \underline{\dot{\overline{\boldsymbol{E}}}}_r^s \cdot \overline{\boldsymbol{x}}^r \right] = \underline{\overline{\boldsymbol{E}}}_r^s \cdot \left[\dot{\overline{\boldsymbol{x}}}^r + \underline{\overline{\tilde{\boldsymbol{\omega}}}}_r^r \cdot \overline{\boldsymbol{x}}^r \right] \tag{2.11}$$

It can be proven (e.g., *Meirovitch*[4, §3.2]) that the tensor

$$\begin{aligned} \underline{\overline{\tilde{\boldsymbol{\omega}}}}_r^r &= \underline{\overline{\boldsymbol{E}}}_s^r \cdot \underline{\dot{\overline{\boldsymbol{E}}}}_r^s \\ \underline{\overline{\tilde{\boldsymbol{\omega}}}}_s^s &= \underline{\overline{\boldsymbol{E}}}_r^s \cdot \underline{\overline{\tilde{\boldsymbol{\omega}}}}_r^r \cdot \underline{\overline{\boldsymbol{E}}}_s^r = \underline{\dot{\overline{\boldsymbol{E}}}}_s^r \cdot \underline{\overline{\boldsymbol{E}}}_s^r \end{aligned} \tag{2.12}$$

is the cross product tensor of angular velocity $\overrightarrow{\boldsymbol{\omega}}$.

We proceed next to obtain $\overrightarrow{\boldsymbol{\omega}}$ as a function of $\overrightarrow{\boldsymbol{a}}$ and ϑ. The following identities can then be verified from the fact that $\overrightarrow{\boldsymbol{a}}$ is a unit vector, ($\underline{\boldsymbol{a}} \cdot \overline{\boldsymbol{a}} = 1$), implying that ($\underline{\boldsymbol{a}} \cdot \dot{\overline{\boldsymbol{a}}} = 0$):

$$\begin{aligned} \underline{\overline{\tilde{\boldsymbol{a}}}} \cdot \underline{\overline{\dot{\tilde{\boldsymbol{a}}}}} \cdot \underline{\overline{\tilde{\boldsymbol{a}}}} &= -(\underline{\boldsymbol{a}} \cdot \dot{\overline{\boldsymbol{a}}}) \, \underline{\overline{\tilde{\boldsymbol{a}}}} \quad = \underline{\overline{\boldsymbol{0}}} \\ \underline{\overline{\tilde{\boldsymbol{a}}}} \cdot \underline{\overline{\tilde{\boldsymbol{a}}}} \cdot \underline{\overline{\dot{\tilde{\boldsymbol{a}}}}} \cdot \underline{\overline{\tilde{\boldsymbol{a}}}} &= -(\underline{\boldsymbol{a}} \cdot \dot{\overline{\boldsymbol{a}}}) \, \underline{\overline{\tilde{\boldsymbol{a}}}} \cdot \underline{\overline{\tilde{\boldsymbol{a}}}} = \underline{\overline{\boldsymbol{0}}} \end{aligned} \tag{2.13}$$

The angular velocity tensor in (2.12), after the differentiation of the transformation matrix (2.2) and algebraic manipulation with the aid of (2.13) and (1.9) is

$$\begin{aligned} \underline{\overline{\tilde{\boldsymbol{\omega}}}}_r^r &= \dot{\vartheta} \, \underline{\overline{\tilde{\boldsymbol{a}}}} + \sin\vartheta \, \underline{\overline{\dot{\tilde{\boldsymbol{a}}}}} - 2\sin^2 \tfrac{\vartheta}{2} \left[\underline{\overline{\tilde{\boldsymbol{a}}}} \cdot \underline{\overline{\dot{\tilde{\boldsymbol{a}}}}} - \underline{\overline{\dot{\tilde{\boldsymbol{a}}}}} \cdot \underline{\overline{\tilde{\boldsymbol{a}}}} \right] \\ &= \dot{\vartheta} \, \underline{\overline{\tilde{\boldsymbol{a}}}} + \sin\vartheta \, \underline{\overline{\dot{\tilde{\boldsymbol{a}}}}} - 2\sin^2 \tfrac{\vartheta}{2} \, \underline{\overline{\widetilde{\underline{\overline{\tilde{\boldsymbol{a}}}} \cdot \dot{\overline{\boldsymbol{a}}}}}} \end{aligned} \tag{2.14}$$

From (2.14), the vector equation for $\overline{\boldsymbol{\omega}}^r$ and $\overline{\boldsymbol{\omega}}^s$ (where the latter can be derived with the same arguments), follows then as

$$\begin{aligned} \overline{\boldsymbol{\omega}}^r &= \dot{\vartheta} \, \overline{\boldsymbol{a}} + \sin\vartheta \, \dot{\overline{\boldsymbol{a}}} - 2\sin^2 \tfrac{\vartheta}{2} \, \underline{\overline{\tilde{\boldsymbol{a}}}} \cdot \dot{\overline{\boldsymbol{a}}} \\ \overline{\boldsymbol{\omega}}^s &= \dot{\vartheta} \, \overline{\boldsymbol{a}} + \sin\vartheta \, \dot{\overline{\boldsymbol{a}}} + 2\sin^2 \tfrac{\vartheta}{2} \, \underline{\overline{\tilde{\boldsymbol{a}}}} \cdot \dot{\overline{\boldsymbol{a}}} \end{aligned} \tag{2.15}$$

The inner or scalar product of (2.15) gives the norm of the angular velocity

$$\omega^2 = \underline{\boldsymbol{\omega}}_r \cdot \overline{\boldsymbol{\omega}}^r = \underline{\boldsymbol{\omega}}_s \cdot \overline{\boldsymbol{\omega}}^s = \dot{\vartheta}^2 + 4\sin \tfrac{\vartheta}{2} \dot{a}^2 \tag{2.16}$$

From (2.15) the time derivative of the rotation angle ϑ is

$$\dot{\vartheta} = \underline{\boldsymbol{a}} \cdot \overline{\boldsymbol{\omega}}^r = \underline{\boldsymbol{a}} \cdot \overline{\boldsymbol{\omega}}^s \tag{2.17}$$

which leads to

$$
\begin{aligned}
\dot{\vartheta}\,\overline{\boldsymbol{a}} &= (\underline{\boldsymbol{a}}\cdot\overline{\boldsymbol{\omega}}^r)\,\overline{\boldsymbol{a}} = \overline{\boldsymbol{\omega}}^r + \underline{\overline{\tilde{\boldsymbol{a}}}}\cdot\underline{\overline{\tilde{\boldsymbol{a}}}}\cdot\overline{\boldsymbol{\omega}}^r \\
&= (\underline{\boldsymbol{a}}\cdot\overline{\boldsymbol{\omega}}^s)\,\overline{\boldsymbol{a}} = \overline{\boldsymbol{\omega}}^s + \underline{\overline{\tilde{\boldsymbol{a}}}}\cdot\underline{\overline{\tilde{\boldsymbol{a}}}}\cdot\overline{\boldsymbol{\omega}}^s
\end{aligned}
\tag{2.18}
$$

At this point it is important to note that in many good reference texts (see for example *Wertz*[5, pp. 511–512]) the authors make the incorrect statement that $\overrightarrow{\boldsymbol{\omega}} = \omega\,\overrightarrow{\boldsymbol{a}}$. Inspection of (2.15) – (2.17) reveals that $\underset{\rightarrow}{\boldsymbol{a}}\cdot\overrightarrow{\boldsymbol{\omega}} = \dot{\vartheta} \neq \omega$. The angular velocity vector $\overrightarrow{\boldsymbol{\omega}}$ is therefore in general not in the direction of the instantaneous rotation axis $\overrightarrow{\boldsymbol{a}}$.

The vector $\dot{\overline{\boldsymbol{a}}}$ can be obtained from (2.15) by the substitution of (2.18) and assuming a solution of the form $[\,\underline{\overline{\boldsymbol{I}}} + \alpha\,\underline{\overline{\tilde{\boldsymbol{a}}}} + \beta\,\underline{\overline{\tilde{\boldsymbol{a}}}}\cdot\underline{\overline{\tilde{\boldsymbol{a}}}}\,]$. With the aid of the identities in (2.13) and (1.9), it leads to

$$
\begin{aligned}
\dot{\overline{\boldsymbol{a}}} &= \tfrac{1}{2}\left[+\underline{\overline{\tilde{\boldsymbol{a}}}} - \cot\tfrac{\vartheta}{2}\,\underline{\overline{\tilde{\boldsymbol{a}}}}\cdot\underline{\overline{\tilde{\boldsymbol{a}}}}\right]\cdot\overline{\boldsymbol{\omega}}^r \equiv \underline{\overline{\boldsymbol{K}}}_r\cdot\overline{\boldsymbol{\omega}}^r \\
&= \tfrac{1}{2}\left[-\underline{\overline{\tilde{\boldsymbol{a}}}} - \cot\tfrac{\vartheta}{2}\,\underline{\overline{\tilde{\boldsymbol{a}}}}\cdot\underline{\overline{\tilde{\boldsymbol{a}}}}\right]\cdot\overline{\boldsymbol{\omega}}^s \equiv \underline{\overline{\boldsymbol{K}}}_s\cdot\overline{\boldsymbol{\omega}}^s
\end{aligned}
\tag{2.19}
$$

Note the notation in (2.19) for $\underline{\overline{\boldsymbol{K}}}_r$. It is a tensor in a mixed base (see *Hassenpflug*[1]), because $\overline{\boldsymbol{a}}^r = \overline{\boldsymbol{a}}^s$. For the transformation between bases it can also be confirmed that

$$
\underline{\overline{\boldsymbol{K}}}_r = \underline{\overline{\boldsymbol{K}}}_s\cdot\underline{\overline{\boldsymbol{E}}}{}_r^s
\tag{2.20}
$$

The general kinematic equations for a rotating base are given by (2.17) and (2.19). The four scalar equations describe only three degrees of freedom and are constrained by $\|\overline{\boldsymbol{a}}\| = 1$. These equations can be integrated to obtain $\underline{\overline{\boldsymbol{E}}}{}_r^s$ as a function of time, but (2.19) is singular for values of $\vartheta = 0, \pm 2\pi, \cdots$. This renders a general numeric solution impractical.

The equations for the angular velocity, (2.15) are well-known, see for example *Shabana*[6, §5.14]. The author could not find any reference to the inverse form for $\dot{\vartheta}$, (2.17), and $\dot{\overline{\boldsymbol{a}}}$, (2.19), in terms of $\overrightarrow{\boldsymbol{\omega}}$, although it is highly likely that they might exist in the classical literature.

2.3 Attitude determination

The classic problem in rotation kinematics is that the angular velocity cannot be integrated to obtain the orientation of a rotating base, because the integral is dependent on the path of integration. The most basic method to find the orientation of $\underline{\overrightarrow{\boldsymbol{E}}}_r$ as a function of time is to integrate (2.12) directly,

$$
\begin{aligned}
\dot{\underline{\overline{\boldsymbol{E}}}}{}_r^s &= \underline{\overline{\tilde{\boldsymbol{\omega}}}}{}_s^s\cdot\underline{\overline{\boldsymbol{E}}}{}_r^s = \begin{bmatrix}\overline{\boldsymbol{\omega}}^s\times\overline{\boldsymbol{e}}_{r1}^s & \overline{\boldsymbol{\omega}}^s\times\overline{\boldsymbol{e}}_{r2}^s & \overline{\boldsymbol{\omega}}^s\times\overline{\boldsymbol{e}}_{r3}^s\end{bmatrix} \\
\dot{\underline{\overline{\boldsymbol{E}}}}{}_s^r &= -\underline{\overline{\tilde{\boldsymbol{\omega}}}}{}_r^r\cdot\underline{\overline{\boldsymbol{E}}}{}_s^r = -\begin{bmatrix}\overline{\boldsymbol{\omega}}^r\times\overline{\boldsymbol{e}}_{s1}^r & \overline{\boldsymbol{\omega}}^r\times\overline{\boldsymbol{e}}_{s2}^r & \overline{\boldsymbol{\omega}}^r\times\overline{\boldsymbol{e}}_{s3}^r\end{bmatrix}
\end{aligned}
\tag{2.21}
$$

Only two of the vectors need to be integrated. The third vector can be obtained from the cross product ($\overline{\boldsymbol{e}}_1 \times \overline{\boldsymbol{e}}_2 = \overline{\boldsymbol{e}}_3$). This method involves six parameters while there are only three degrees of freedom. With a lot of effort and by careful selection of elements from the orthogonality constraint requirement $\underline{\overline{\boldsymbol{E}}}{}_r^s\cdot\underline{\overline{\boldsymbol{E}}}{}_s^r = \underline{\overline{\boldsymbol{I}}}$, it can be refined to three parameters . It is also advisable that the constraint equation be enforced through frequent normalization, to compensate for the fact that the constraints are not taken into account during integration.

3 Euler symmetric parameters

3.1 Background

Throughout history many parameterization methods were devised to obtain the relationships between the orientation of a rotating base and its angular velocity.

The Euler symmetric parameter method is one of the classic methods. It has gained popularity in the aerospace engineering environment for foolproof attitude determination algorithms, because it contains no numerical singularities. It has the disadvantage that it is a four-parameter method describing three degrees of freedom, and therefore an additional differential equation, together with its constraint, must be solved.

It is also called the rotation quaternion because it can be represented as a unit quaternion, obeying all the rules of quaternion algebra.

3.2 Transformation matrix

After inspection of (2.2), define the four Euler parameters

$$q_0 = \cos\tfrac{\vartheta}{2} \qquad \overline{\boldsymbol{q}} = \sin\tfrac{\vartheta}{2}\,\overline{\boldsymbol{a}} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{3.1}$$

The transformation matrix (2.2), in terms of the Euler parameters, is then

$$\underline{\overline{\boldsymbol{E}}}_r^s(q_0, \overline{\boldsymbol{q}}) = \underline{\overline{\boldsymbol{I}}} + 2q_0\,\underline{\overline{\tilde{\boldsymbol{q}}}} + 2\,\underline{\overline{\tilde{\boldsymbol{q}}}}\cdot\underline{\overline{\tilde{\boldsymbol{q}}}} \tag{3.2}$$

$$\underline{\overline{\boldsymbol{E}}}_s^r(q_0, \overline{\boldsymbol{q}}) = \underline{\overline{\boldsymbol{E}}}_r^s(q_0, -\overline{\boldsymbol{q}}) \tag{3.3}$$

or in element form

$$\underline{\overline{\boldsymbol{E}}}_r^s(q_0, \overline{\boldsymbol{q}}) = \begin{bmatrix} 2q_0^2 + 2q_1^2 - 1 & 2(q_1q_2 - q_3q_0) & 2(q_1q_3 + q_2q_0) \\ 2(q_1q_2 + q_3q_0) & 2q_0^2 + 2q_2^2 - 1 & 2(q_2q_3 - q_1q_0) \\ 2(q_1q_3 - q_2q_0) & 2(q_2q_3 + q_1q_0) & 2q_0^2 + 2q_3^2 - 1 \end{bmatrix} \tag{3.4}$$

The four Euler parameters are not independent, but are constrained by the condition for the transformation matrix, $\underline{\overline{\boldsymbol{E}}}_r^s\cdot\underline{\overline{\boldsymbol{E}}}_s^r = \underline{\overline{\boldsymbol{I}}}$, which implies that

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = q_0^2 + \underline{\boldsymbol{q}}\cdot\overline{\boldsymbol{q}} = 1 \tag{3.5}$$

and which is indeed satisfied by (3.1).

From (3.2) it is clear that changing the signs of all the Euler parameters simultaneously does not affect the transformation matrix

$$\underline{\overline{\boldsymbol{E}}}_r^s(-q_0, -\overline{\boldsymbol{q}}) = \underline{\overline{\boldsymbol{E}}}_r^s(q_0, \overline{\boldsymbol{q}}) \tag{3.6}$$

The initial values of q_0 and $\overline{\boldsymbol{q}}$ can be obtained for a known transformation matrix $\underline{\overline{\boldsymbol{E}}}_r^s = E_{ij}$ from (3.4). The following equations are the relationships that can be deducted

$$\begin{aligned} 4q_0^2 &= 1 + E_{11} + E_{22} + e_{33} \\ 4q_1^2 &= 1 + E_{11} - E_{22} - e_{33} \\ 4q_2^2 &= 1 - E_{11} + E_{22} - E_{33} \\ 4q_3^2 &= 1 - E_{11} - E_{22} + E_{33} \end{aligned} \tag{3.7}$$

$$
\begin{aligned}
4q_1q_0 &= E_{32} - E_{23} \\
4q_2q_0 &= E_{13} - E_{31} \\
4q_3q_0 &= E_{21} - E_{12}
\end{aligned}
\qquad \text{and} \qquad
\begin{aligned}
4q_1q_2 &= E_{12} + E_{21} \\
4q_1q_3 &= E_{13} + E_{31} \\
4q_2q_3 &= E_{23} + E_{32}
\end{aligned}
\tag{3.8}
$$

The absolute values of Euler parameters are obtained from (3.7).

$$
\begin{aligned}
|2\,q_0| &= \sqrt{1 + E_{11} + E_{22} + E_{33}} \\
|2\,q_1| &= \sqrt{1 + E_{11} - E_{22} - E_{33}} \\
|2\,q_2| &= \sqrt{1 - E_{11} + E_{22} - E_{33}} \\
|2\,q_3| &= \sqrt{1 - E_{11} - E_{22} + E_{33}}
\end{aligned}
\tag{3.9}
$$

The unity constraint (3.5), implies that at least one of the Euler parameters is not zero. Furthermore, a simultaneous sign change of all the Euler parameters has no effect on the transformation matrix, see (3.6). To avoid singularities and for the best numerical accuracy, select the absolute value of the largest parameter from (3.9) as initial value and then calculate the Euler parameters accordingly from (3.7) and (3.8).

$$
\begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}
=
\begin{bmatrix} \frac{|2\,q_0|}{2} \\ \frac{E_{32}-E_{23}}{2\,|2\,q_0|} \\ \frac{E_{13}-E_{31}}{2\,|2\,q_0|} \\ \frac{E_{21}-E_{12}}{2\,|2\,q_0|} \end{bmatrix}
\text{ or }
\begin{bmatrix} \frac{E_{32}-E_{23}}{2\,|2\,q_1|} \\ \frac{|2\,q_1|}{2} \\ \frac{E_{12}+E_{21}}{2\,|2\,q_1|} \\ \frac{E_{13}+E_{31}}{2\,|2\,q_1|} \end{bmatrix}
\text{ or }
\begin{bmatrix} \frac{E_{13}-E_{31}}{2\,|2\,q_2|} \\ \frac{E_{12}+E_{21}}{2\,|2\,q_2|} \\ \frac{|2\,q_2|}{2} \\ \frac{E_{23}+E_{32}}{2\,|2\,q_2|} \end{bmatrix}
\text{ or }
\begin{bmatrix} \frac{E_{21}-E_{12}}{2\,|2\,q_3|} \\ \frac{E_{13}+E_{31}}{2\,|2\,q_3|} \\ \frac{E_{23}+E_{32}}{2\,|2\,q_3|} \\ \frac{|2\,q_3|}{2} \end{bmatrix}
\tag{3.10}
$$

3.3 Time derivatives of the Euler parameters

The time derivatives of the Euler parameters (3.1), with the aid of (2.17) and (2.19), are for $\vec{\boldsymbol{\omega}}$ in terms of base $\underline{\vec{\boldsymbol{E}}}_r$

$$
\begin{aligned}
\dot{q}_0 &= -\tfrac{1}{2}\sin\tfrac{\vartheta}{2}\;\dot{\vartheta} \\
&= -\tfrac{1}{2}\sin\tfrac{\vartheta}{2}\;\underline{\boldsymbol{a}}\cdot\overline{\boldsymbol{\omega}}^r \\
&= -\tfrac{1}{2}\;\underline{\boldsymbol{q}}\cdot\overline{\boldsymbol{\omega}}^r
\end{aligned}
\qquad\qquad
\begin{aligned}
\dot{\overline{\boldsymbol{q}}} &= \tfrac{1}{2}\cos\tfrac{\vartheta}{2}\;\dot{\vartheta}\;\overline{\boldsymbol{a}} + \sin\tfrac{\vartheta}{2}\;\dot{\overline{\boldsymbol{a}}} \\
&= \tfrac{1}{2}\cos\tfrac{\vartheta}{2}\;\overline{\boldsymbol{\omega}}^r + \tfrac{1}{2}\sin\tfrac{\vartheta}{2}\;\underline{\overline{\widetilde{\boldsymbol{a}}}}\cdot\overline{\boldsymbol{\omega}}^r \\
&= \tfrac{1}{2}\;q_0\;\overline{\boldsymbol{\omega}}^r + \tfrac{1}{2}\;\underline{\overline{\widetilde{\boldsymbol{q}}}}\cdot\overline{\boldsymbol{\omega}}^r
\end{aligned}
\tag{3.11}
$$

The same procedure can be repeated for $\vec{\boldsymbol{\omega}}$ in terms of base $\underline{\vec{\boldsymbol{E}}}_s$. Equation (3.11) can be rewritten in the more familiar matrix format

$$
\begin{bmatrix} \dot{q}_0 \\ \dot{\overline{\boldsymbol{q}}} \end{bmatrix}
= \frac{1}{2}
\begin{bmatrix} 0 & -\underline{\boldsymbol{\omega}}_r \\ \overline{\boldsymbol{\omega}}^r & -\underline{\overline{\widetilde{\boldsymbol{\omega}}}}^r_r \end{bmatrix}
\cdot
\begin{bmatrix} q_0 \\ \overline{\boldsymbol{q}} \end{bmatrix}
\quad = \quad
\frac{1}{2}
\begin{bmatrix} 0 & -\underline{\boldsymbol{\omega}}_s \\ \overline{\boldsymbol{\omega}}^s & +\underline{\overline{\widetilde{\boldsymbol{\omega}}}}^s_s \end{bmatrix}
\cdot
\begin{bmatrix} q_0 \\ \overline{\boldsymbol{q}} \end{bmatrix}
\tag{3.12}
$$

The constraint equation (3.5) in differential form is

$$
q_0\dot{q}_0 + q_1\dot{q}_1 + q_2\dot{q}_2 + q_3\dot{q}_3 = \begin{bmatrix} q_0 & \underline{\boldsymbol{q}} \end{bmatrix} \cdot \begin{bmatrix} \dot{q}_0 \\ \dot{\overline{\boldsymbol{q}}} \end{bmatrix} = 0
\tag{3.13}
$$

If (3.12) is substituted into (3.13), it confirms, as expected, that (3.12) still satisfies the constraint condition.

3.4 Notes on numerical integration

Equation (3.12) in general cannot be integrated analytically and we must resort to numerical integration methods. For illustration purposes, consider the simplest numerical integration scheme, namely the first order Euler method. Let

$$\overline{\boldsymbol{Q}} = \begin{bmatrix} q_0 \\ \overline{\boldsymbol{q}} \end{bmatrix} \qquad \text{and} \qquad \dot{\overline{\boldsymbol{Q}}} = \begin{bmatrix} \dot{q}_0 \\ \dot{\overline{\boldsymbol{q}}} \end{bmatrix} = \tfrac{1}{2}\, \underline{\overline{\boldsymbol{\Omega}}_r} \cdot \overline{\boldsymbol{Q}} = \tfrac{1}{2}\, \underline{\overline{\boldsymbol{\Omega}}_s} \cdot \overline{\boldsymbol{Q}} \tag{3.14}$$

with $\underline{\overline{\boldsymbol{\Omega}}_r}$, $\underline{\overline{\boldsymbol{\Omega}}_r} \in \mathbb{R}^{4\times 4}$ from (3.12)

$$\underline{\overline{\boldsymbol{\Omega}}_r} = \begin{bmatrix} 0 & -\underline{\boldsymbol{\omega}}_r \\ \overline{\boldsymbol{\omega}}^r & -\underline{\overline{\overline{\boldsymbol{\omega}}}}{}_r^r \end{bmatrix} \qquad \text{and} \qquad \underline{\overline{\boldsymbol{\Omega}}_s} = \begin{bmatrix} 0 & -\underline{\boldsymbol{\omega}}_s \\ \overline{\boldsymbol{\omega}}^s & +\underline{\overline{\overline{\boldsymbol{\omega}}}}{}_s^s \end{bmatrix} \tag{3.15}$$

The Euler parameters at time t can then be updated over a time step Δt with

$$\overline{\boldsymbol{Q}}(t{+}\Delta t) \approx \overline{\boldsymbol{Q}}(t) + \Delta t\, \dot{\overline{\boldsymbol{Q}}}(t) = \left[\, \underline{\overline{\boldsymbol{I}}} + \Delta t\, \underline{\overline{\boldsymbol{\Omega}}}(t) \right] \cdot \overline{\boldsymbol{Q}}(t) \tag{3.16}$$

where $\underline{\overline{\boldsymbol{\Omega}}}$ is either $\underline{\overline{\boldsymbol{\Omega}}_r}$ or $\underline{\overline{\boldsymbol{\Omega}}_s}$.

Assume that $\overline{\boldsymbol{Q}}(t)$ conforms to the constraint condition, then after the integration time step it is found that

$$\underline{\boldsymbol{Q}}^{\mathsf{T}}(t{+}\Delta t) \cdot \overline{\boldsymbol{Q}}(t{+}\Delta t) = 1 + \tfrac{1}{4} \left(\Delta t\, \omega(t) \right)^2 \tag{3.17}$$

It is thus clear that the integration process results in an updated set of parameters that violates the required constraint. This condition only vanishes in the limit when $\Delta t{\to}0$. This necessitates that the parameters be normalized at regular intervals for most numerical integration methods or that the integration method be tailored to take the constraint into consideration.

Wertz[5, §17.1] discusses a useful approximate integrator of the Euler parameters kinematic equations, by *Wilcox*[7] and *Iwens & Farrenkopf*[8]. This integration method is used for realtime onboard attitude determination of spacecraft from gyro telemetry.

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